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Recurrent Riemann–Cartan Spacetimes

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An example of a Riemann recurrent spacetime with torsion is given. The implications for the Segré classification are investigated. It is also shown that the recurrent vector coincides with the torsion vector in the case where the Ricci recurrent spacetime is an Einstein space.

1. INTRODUCTION

Several authors (Thompson, 1969; Walker, 1950; Ruse, 1946) have investigated recurrent Riemannian manifolds in detail. Applications to general relativity, especially to gravitational pp waves, have been considered by Thompson (1968) and Sciama (1961). More recently Hall (1976, 1977) has applied the theory of Ricci recurrent spacetimes to the study of the classification of the Weyl tensor (Petrov, 1969) and to the Segré classification of the Ricci tensor. In this paper I introduce the idea of Riemann-Cartan recurrent spaces, constructing a spacetime with torsion which admits a recurrence vector which coincides with the torsion vector in the case where the Ricci tensor represents an Einstein space (Petrov, 1969). It is also shown that the Segré type of the recurrent spacetime depends on the torsion vector.

2. THE MAIN RESULT

In this section I shall prove the following theorem.

Theorem. Consider a Riemann-Cartan manifold of Lorentz signature $(-+++)$ where the Riemann tensor takes the form

$$R_{klmn} = \lambda\,(g_{kn}g_{ml} - g_{km}g_{nl}) \tag{1}$$

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i.e., an Einstein space, with Ricci tensor $R_{ij} = \lambda g_{ij}$ in U_4. The torsion is given by

$$S^k_{ij} = \tfrac{2}{3}\delta^k_{[j} S_{i]} \tag{2}$$

where the vector torsion S_i is considered to be the gradient of a scalar function S ($S_i = S_{,i}$) (Buchdahl, 1989). Here the comma indicates the partial derivative as usual and the semicolon the covariant derivative in terms of the Riemann-Cartan connection. Let us also suppose that the Riemann tensor in U_4 is recurrent, i.e., obeys the relation

$$R_{ijkl;n} = R_{ijkl}\xi_n \tag{3}$$

Then the recurrence vector ξ_n is proportional to the torsion vector of spacetime.

Proof. Since $g_{ij;k} = 0$ we can write, from equation (1),

$$R_{plmn;k} = \lambda_{,k}(g_{pn}g_{ml} - g_{pm}g_{nl}) \tag{4}$$

Comparison of equations (3) and (4) yields

$$\lambda_{,k} = \lambda\xi_k \tag{5}$$

which is equivalent to $\xi_k = (\ln\lambda)_{,k}$. However, from the differential relation (Schouten, 1954)

$$R^m_{i[jk;l]} = -2S^n_{[jk}R^m_{|i|l]n} \tag{6}$$

one finds that the torsion must be a vector of the type $S_k = \frac{3}{4}\ (\ln\lambda)_{,k}$. Comparison of this relation to ξ_k above completes the proof. ■

One natural way to obtain a Riemann-Cartan spacetime where the Riemann tensor is covariant constant in this case is to make the torsion S_i vanish, but this is the trivial Riemannian case.

By expression (5) one immediately observes that the vanishing of the torsion vector implies the constancy of λ, which by (1) means that the space has constant curvature (Kramer *et al.*, 1980).

A more elegant proof of the above theorem is to substitute equation (2) into (6), obtaining the relation

$$R^m_{i[jk;l]} = -\tfrac{4}{3}S_{[j}R^m_{|i|lk]}$$

which is compatible with a natural recurrent relation $R^m_{ijk;l} = R^m_{ijk}\xi_l$. In this sense one may say that the Riemann-Cartan spacetimes are naturally recurrent.

The above theorem may have interesting implications for the study of electromagnetism in spacetimes with torsion, since it is known that the electric and magnetic fields can be constructed from the torsion vector (Smalley, 1986; Gogala, 1980).

From the point of view of the Segré classification in U_4 (Garcia de Andrade, 1989*a*,*b*), one can observe that the expression (1) implies by contraction

$$R_{ij} = \lambda g_{ij} \tag{7}$$

which represents the Ricci tensor of Einstein spaces (Petrov, 1969). Contraction of (7) with the null tetrad leg l^k yields

$$R_{ik} l^k = \lambda l_i \tag{8}$$

which means that the Ricci tensor in U_4 admits an eigenvector l^k with corresponding eigenvalue proportional to the torsion vector. Now suppose that R_{ij} is a Ricci recurrent tensor (Hall, 1976); then

$$R_{ij;k} = R_{ij}\zeta_k \tag{9}$$

From (7) and (9) we obtain the eigenvalue $\lambda = c\exp(\int S_i\, dx^i)$ or $\lambda = e^S$ in the case the torsion is scalar.

The form of this eigenvalue can be connected with holonomy groups (Hall and Kay, 1988) in spacetimes with torsion. Considering a real null tetrad (l, n, x, y) obeying the orthonormality relations $l \cdot n = -1$, $xx = yy = 1$, other inner products being zero, it is easy to show, by contraction of (7) with x^l and y^l, that there are two spacelike eigenvectors with common eigenvalue λ. This shows that the Segré type of the Ricci tensor is $((1, 1, 1, 1))$, where the interior parentheses represent the repeated eigenvalues. In the case of torsion-free Riemannian spacetime, the Ricci recurrent spacetime in U_4 reduces to Hall's result in GR (Hall, 1977). It is also interesting to point out that in the case of Segré type $(1, 1, 1, 1)$ the Bianchi identities in U_4

$$t_{i;[jk]} = R_{ijkl}t^l - 2S_{jk}^{m}t_{i;m} \tag{10}$$

for the covector t_i do not lead to the result $R_{ijkl}K^l = 0$ as in the case of radiative gravitational recurrent spacetimes considered by Sciama (1961). This is due to the fact that there is no Petrov type N solution in Riemann–Cartan spacetimes, since there is no propagation of torsion in the Einstein–Cartan theory. A more detailed investigation of the general recurrent spacetimes with torsion will appear elsewhere.

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